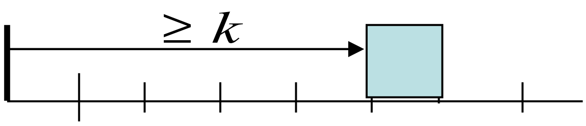
Dr. Wilhelm, Claus: symmetric binary channel with memory;
A-Model, especial suitable for M2M communication; examples for beginners and students; to test with Excel;
For background see downloads!

Example 1

Given the **“wireless short-packet (WSP) protocol”** ISO/IEC 14543-3-10 for M2M (Machine to Machine) communication, 125 kbit/s data rate, 868,3 MHz.
Each frame is transmitted three times in one direction without feedback.
The Header starts with 8 PRE-bits for synchronisation followed by 4 SOF-bits as the start of synchronisation: 101010101001
If in these 12 bits one or more errors occur the frame cannot start.
Find the probability u(k) for the next error bit which follows after k = 12 or more error-free bit’s.
Also the probability that the frame start is wrong or not is: $1-u(k=12)$



$$u(0)=1; u(1)=\alpha e^{-\beta}; \quad \text{where } e^{-\beta}=(1-p_e^{1/\alpha});$$

with bit error probability $p_e < 0.01$

and in practice with bundling factor $(1-\alpha)$; $0.5 < \alpha < 0.95$

(for an BSC-Channel, without memory is $(1-\alpha)=0$;)

Using Excel, the follwing can be recursively calculated:

$$u(k+1)=u(k)\frac{k+\alpha}{k+1}e^{-\beta}$$

given:

a) pe = 0.001; alpha = 0.9

b) pe = 0.01; alpha = 0.7 (strong bundled errors)

result:

$$1-u(k=12; pe=0.001; \alpha=0.9)=0.27685541$$

$$1-u(k=12; pe=0.01; \alpha=0.7)=0.66283848$$

$$0.66283848^3=0.2912213$$

$$P(non-start; 3times)=0.66283848^3=29\%$$

Even with triple transfer, in 29% all of experiments a non-start occurred.

Example 2

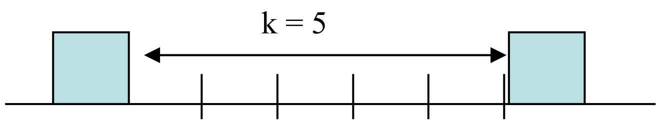
What is the probability v(k) that after the last error bit a gap follows with a length of

a) k = 0 bits (double error);

b) k = 5 bits

c) k = 48 bits ?

with pe = 0.001; alpha = 0.9



where

$$v(k) = u(k) - u(k+1); u(0) = 1;$$

similar to example 1, the results can be calculated recursively.

the results:

v(0) = 0.10041774
v(5) = 0.01349111
v(48) = 0.00155455

Example 3

What is the probability $p_b(n)$ that a block of length n has errors ?

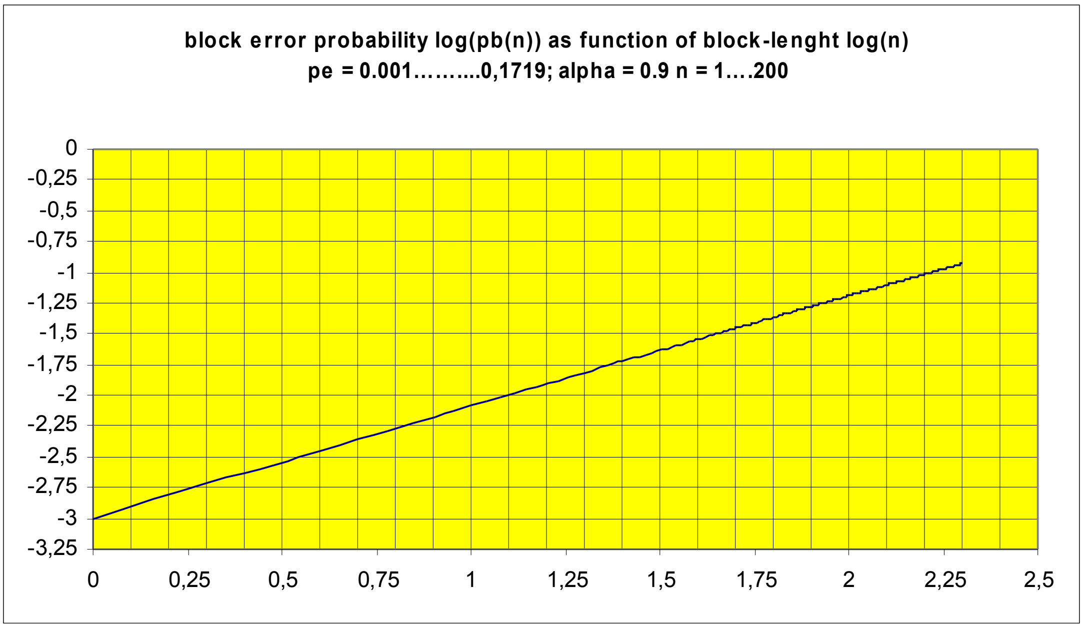
with pe = 0.001; alpha = 0.9 ; $e^{-\beta} = 0.99953584$

$$p_b(n) = p_e \sum_{k=0}^{n-1} u(k); \quad u(k+1) = u(k) \frac{k+\alpha}{k+1} e^{-\beta}; u(0) = 1; u(1) = \alpha e^{-\beta};$$

$$p_b(n+1) = p_b(n) + p_e \cdot u(n);$$

$$p_b(1) = p_e$$

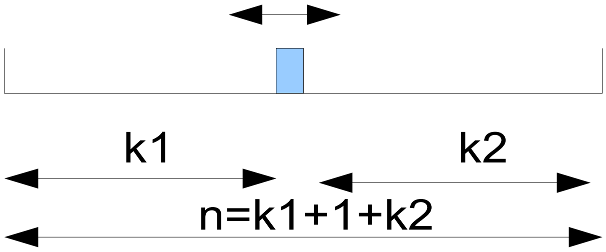
the results: *pb(1) = 0.001; pb(5) = 0.00438348; pb(48) = 0.03350905*



Example 4

We are looking for the probability of all single errors in a block of length *n*.

The sum of probabilities of ***n*** different patterns:



$$p(n+1;g=1)=p(n;g=1)\cdot\frac{(2\alpha+n-1)}{n}\cdot e^{-\beta}$$

with $p(n=1;g=1)=p_e=0.001;\ \alpha=0.9$

the results:

p(5;g =1) = 0.00382320
p(31; g =1) = 0.01647384

Example 5
Linear block code CCITT CRC 16

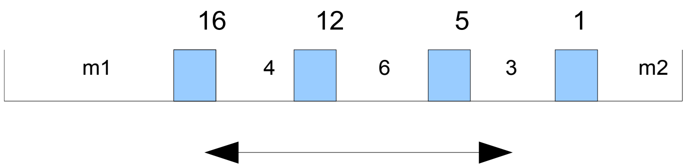
Given a ***(48;16)*** block code with block length ***n*** and ***k = 16*** control bits and ***32*** information bits.

The generator polynomial is

$$x^{16}+x^{12}+x^{5}+1$$

A false block will go undetected if the error structure is identical with the above polynomial. There are 31 positions which have the shortest undetected erroneous structures.

Total, there are $2^{16}=65536$ such undetected structures.



$$P(structureCRC16;n=48)=p(31;g=1)\cdot v(4)\cdot v(6)\cdot v(3)$$

There are 31 possible pattern in this block of length 48;
with pe = 0.001; alpha = 0.9

the result is the product of four factors:

P(31;g=1) = 0.01345740
v(3) = 0.02100726
v(4) = 0.01645272
v(6) = 0.01141715

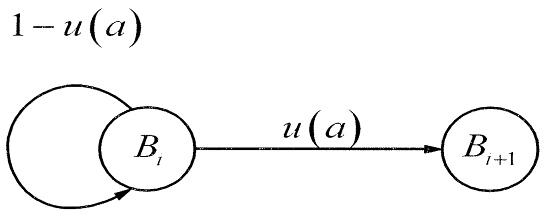
therefore **5.3103867E-8;**

and the probability of erroneous blocks is

pb(48) = **3.350905E-2** (see example 3);

Example 6
Distributions of numbers of errors in bursts:

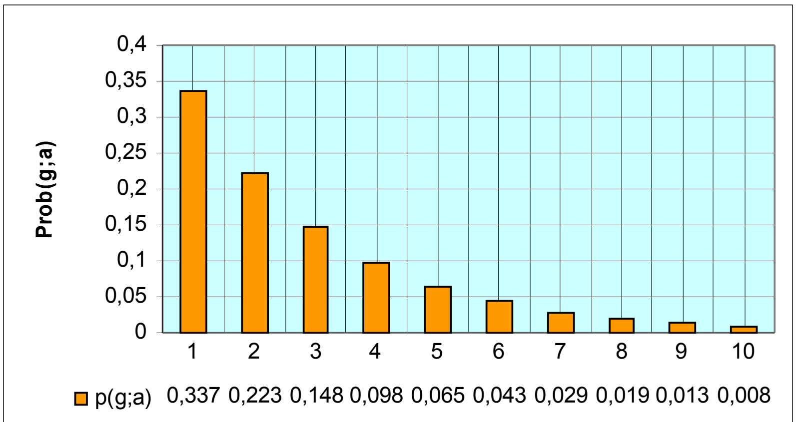
A burst begins with a bit error and ends when a consecutive error-free bits follow.
The number of erroneous bits in the burst comprises the burst-weight **g**.
It is plausible that when a is equal to 1 the weight **g** is equal to l
Calculate the probabilities for **1, 2, 3, 4, 5, and 20** .errors in the burst.
Given $p_e=0.01;\alpha=0.7;a=12$



$$P(g;a) = u(a)\cdot\left[1- u(a)\right]^{g-1}$$

$u(a = 12)$ is calculated recursively as in **Example 1**

$u(12) = 1- 0.66283848 = 0.33716152$



Example 6 : g errors, burst distance a=12, pe=0,01; alpha=0,7

results:

$P(g = 1;12) = 0.33716152; P(g = 2;12) = 0.22348363;$

$P(g= 3;12)= 0.14813355; P(g= 4;12)= 0.09818862$

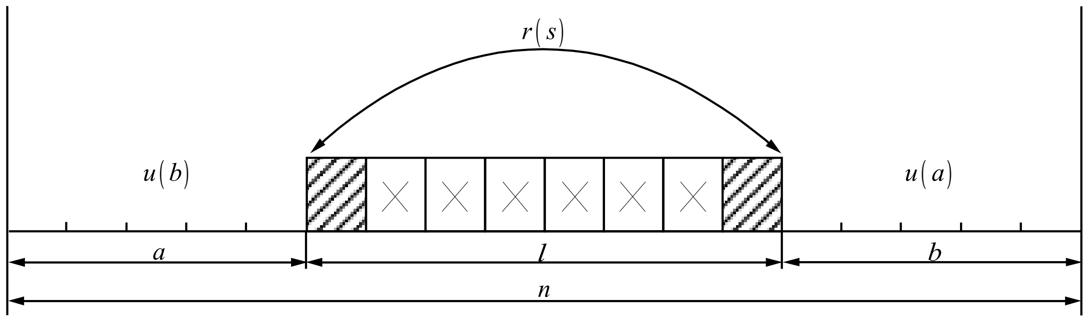
$P(g= 5;12)= 0.06508319; P(g= 20;12)= 0.0001632834$

Example 7

Distribution $p(l;n)$ of burst length l in blocks of length *n = 5* bits

where l = 1,2, 3,….***n;***   bit error probability $p_e = 0.0008488$;

bundling factor $(1- \alpha) = 0.16; \alpha = 0.84$; and $e^{-\beta} = (1- p_e^{1/\alpha}) = 0.9997792931\cdot$



The formula is derived in downloads: *„channel model part 1“*   (73)

$$p\left(l;n\right) = p_{\mathrm{e}} \frac{2\alpha \left(2\alpha + 1\right)\cdots\left(2\alpha + n - l - 1\right)}{\left(n - l\right)!} \mathrm{e}^{-\beta\left(n-l\right)}$$

$$\times \left(1 - \frac{\alpha}{1!}\mathrm{e}^{-\beta} - \frac{\alpha\left(1-\alpha\right)}{2!}\mathrm{e}^{-2\beta} - \cdots - \frac{\alpha\left(1-\alpha\right)\cdots\left(l-2-\alpha\right)}{\left(l-1\right)!}\mathrm{e}^{-\left(l-1\right)\beta}\right).$$

For calculation (73), it is better to make the following transformation

$$p(l,n) = p_u\left(n - l + 1\right) \cdot r_s\left(l - 1\right)$$

and calculate each of these factors recursively.

Where $p_u(n - l + 1)$ is the probability of a single error in the block of length $n = a + b + l$ (see Example 4).

The factor $r_s(l - 1)$ is the probability of a burst of length $\boldsymbol{l}$.

To prepare for programming in higher languages ,such as ***Matlab or C++****, the following provide test-solutions, programmable in Excel ,which are helpful in understanding the method used in the solutions.*

Next we calculate recursively:

$$p_u(l = 5; n = 5) = p_e$$

$$p_u(l = 4; n = 5) = p_u(5;5) \cdot (2\alpha / 1) \cdot e^{-\beta}$$

$$p_u(l = 3; n = 5) = p_u(4;5) \cdot ((2\alpha + 1) / 2) \cdot e^{-\beta}$$

$$p_u(l = 2; n = 5) = p_u(3;5) \cdot ((2\alpha + 2) / 3) \cdot e^{-\beta}$$

$$p_u(l = 1; n = 5) = p_u(2;5) \cdot ((2\alpha + 3) / 4) \cdot e^{-\beta}$$

and also

$$r_s(l = 1) = 1$$

$$r_s(l = 2) = r_s(l = 1) - (\alpha / 1!) \cdot e^{-\beta}$$

$$r_s(l = 3) = r_s(l = 2) - ((\alpha \cdot (1 - \alpha) / 2!) \cdot e^{-2\beta}$$

$$r_s(l = 4) = r_s(l = 3) - ((\alpha \cdot (1 - \alpha) \cdot (2 - \alpha) / 3!) \cdot e^{-3\beta}$$

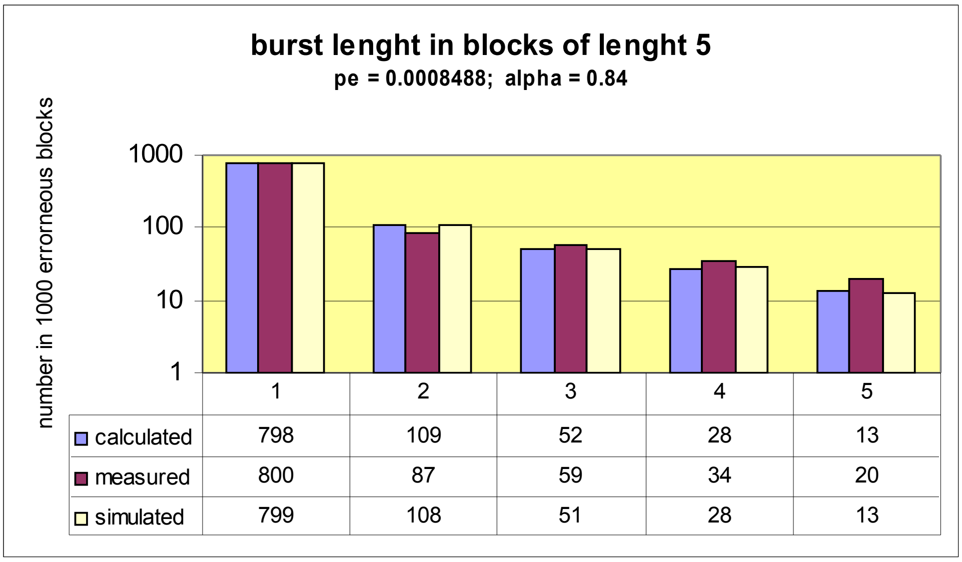
$$r_s(l = 5) = r_s(l = 4) - ((\alpha \cdot (1 - \alpha) \cdot (2 - \alpha) \cdot (3 - \alpha) / 4!) \cdot e^{-4\beta}$$

The result calculated with Excel:

pe=0.0008488	alpha=0.84	n=5		pb(5)=		
length l	pu	rs	p(l;n=5) = pu*ps	**3,4334588506E-03**	measured	simulated
				10^3*p(l;n=5)/pb(5)		
1	0,0027399865	1,0000000000	2,7399865000E-03	798	800	799
2	0,0023423858	0,1601853938	3,7521599180E-04	109	87	108
3	0,0019099752	0,0930150535	1,7765644541E-04	52	59	51
4	0,0014256693	0,0670482543	9,5588637774E-05	28	34	28
5	0,0008488000	0,0530292775	4,5011250742E-05	13	20	13
		Summe=pb(5)	3,4334588257E-03	1000	1000	999

It can be seen that the larger length occurs more frequently because in this model a memory does not proceed beyond the last error bit. On the other hand, the χ^2 *-Test shows no significant differenz.*

But in comparing codes or procedures this A-Model is sufficient and easy to use in most cases. The advantage is, having the first easy method to calculate sums of events



Example 8

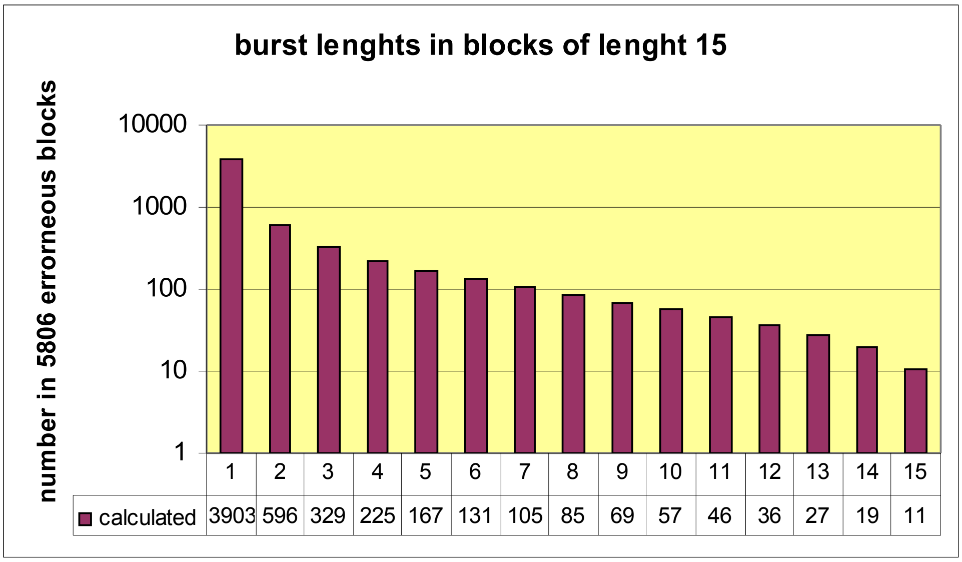
Distribution $p(l;n)$ of burst length l in blocks of length *n* = *15* bits

where l = 1,2, 3,….***n;*** bit error probability $p_e = 0.0008; \alpha = 0.8; e^{-\beta} = 0.9998548645$

On can it recursively develop in the same way as in Example 7 from the basic Formula (73).

The result calculated with Excel:

pe=0.00085		pb(15)=		5806 err.blocks
length l	p(l;n=15)	**7,9153586826E-03**	calculated	simulated
		p(l;n=15)/pb(15)	5806*p(l;n=15)/pb(15)	
1	0,0047821493	0,6041607831	3508	3469
2	0,0009177902	0,1159505509	673	750
3	0,0005267615	0,0665492899	386	355
4	0,0003681637	0,0465125732	270	293
5	0,0002794990	0,0353109709	205	195
6	0,0002216502	0,0280025466	163	168
7	0,0001802212	0,0227685450	132	128
8	0,0001485954	0,0187730469	109	111
9	0,0001232668	0,0155731161	90	267
10	0,0001021736	0,0129082716	75	(sum of 9+10+11+12)
11	8,3988972658E-05	0,0106108865	62	
12	6,7771249004E-05	0,0085619934	50	
13	5,2749022590E-05	0,0066641355	39	70
14	3,8106656360E-05	0,0048142678	28	(sum of 13+14+15)
15	2,2471904462E-05	0,0028390254	16	
Sum total	7,9153587051E-03	1,0000000028E+00	5806	5806


burst lenghts in blocks of lenght 15
number in 5806 errorneous blocks
10000
1000
100
10
1
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
calculated 3903 596 329 225 167 131 105 85 69 57 46 36 27 19 11